Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 1, pp. 122–128, 2008

BUCKLING ANALYSIS OF AN ELASTIC BAR USING THE BUBNOV–GALERKIN METHOD

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UDC 539.3

The factors responsible for the errors often encountered in the stability analysis of elastic systems are found by testing the Bubnov–Galerkin method for the buckling problem of a single-layer flexible elastic bar. Refined formulas are obtained for the maximum deflection of a longitudinally compressed hinged three-layer bar.

Key words: *Bubnov–Galerkin method, elastic bar, buckling, buckling shapes, maximum deflection.*

Introduction. The exact relation between a longitudinal compressive load P acting on an elastic hinged bar of length l and the deflection f was obtained by Euler in the form of the complete elliptic integral of the first kind [1, p. 441]

$$l = 2\sqrt{\frac{EI}{P}} \int\limits_0^{\pi/2} \frac{d\psi}{\sqrt{1 - (f^2P/(4EI))\sin^2\psi}}$$

and in the form of the series

$$l = \pi\sqrt{\frac{EI}{P}}\Big\{1 + \Big(\frac{1}{2}\Big)^2\frac{f^2P}{4EI} + \Big(\frac{1\cdot 3}{2\cdot 4}\Big)^2\Big(\frac{f^2P}{4EI}\Big)^2 + \Big(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\Big)^2\Big(\frac{f^2P}{4EI}\Big)^3 + \ldots\Big\}, \qquad (1)$$

where EI is the flexural rigidity of the bar. The postcritical equilibrium shapes of compressed bars were first studied by Lagrange, and then, using tables of elliptic integrals, by Krylov [2] and Popov [3]. However, the methods described in [2, 3] are labor-consuming for practical use because of the necessity of calculating the parameters expressed in terms of elliptic integrals of the first and second kind. This is one of the factors motivating the development of methods (solution of the linearized differential equations corresponding to the original nonlinear equations; approximation of the elliptic integrals describing exact solutions, etc.) for deriving approximate formulas for the bar deflections and buckling modes. Another reason for addressing the problem of determining the deflection curve shape is that, because of the existence of the exact solution, it is used as a test problem in constructing effective approximate solutions.

Mises Linearization of the Differential Equation of the Deflection Curve. Mises [4] proposed the following method of deriving approximate formulas for bar deflections as a function of load. The bar deflection curve can be represented by the differential equation

$$\frac{d^2y}{ds^2} + \frac{P}{EI}\, y\cos\theta = 0, \qquad (2)$$

where θ is the angle between the tangent and the Ox axis (the originally rectilinear axis of the bar); s is the arc length of the deflection curve. Assuming, as a first approximation, that the deflection curve has the shape of the sinusoid

$$y = c\sin(\pi s/l) \qquad (3)$$

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and taking into account the equality $\sin\theta = dy/ds = c\pi\cos(\pi s/l)/l$ and that the quantity c is small compared to l, we have

$$\cos\theta = \sqrt{1 - \Big(\frac{c\pi}{l}\cos\frac{\pi s}{l}\Big)^2} \approx 1 - \frac{\pi^2 c^2}{2l^2}\cos^2\frac{\pi s}{l}. \tag{4}$$

Substitution of expressions (3) and (4) into Eq. (2) yields

$$\frac{d^2y}{ds^2} + \frac{P}{EI}\,y = \frac{\pi^2 c^3}{2l^2}\,\frac{P}{EI}\,\cos^2\frac{\pi s}{l}\,\sin\frac{\pi s}{l}. \tag{5}$$

Replacing the quantity P/EI on the right side of (5) by the close quantity π^2/l^2 and performing trigonometric transformations, we obtain

$$\frac{d^2y}{ds^2} + \frac{P}{EI}\,y = \frac{\pi^4 c^3}{8l^4}\Big(\sin\frac{\pi s}{l} + \sin\frac{3\pi s}{l}\Big). \tag{6}$$

We find the integral of Eq. (6) that satisfies the boundary conditions $y(0) = y(l) = 0$:

$$y = c\sin(\pi s/l) + c_1\sin(3\pi s/l).$$

Here $c^2 = 8l^2(P/P_* - 1)/\pi^2$, where $P_* = EI(\pi/l)^2$ is the Euler critical load. In view of the approximate equality $P/EI \approx (\pi/l)^2$, $c_1 \approx -(\pi/l)^2 c^3/64$ for $P \approx P_*$. We note that in [5] the same relation between the coefficients c and c_1 was obtained by a perturbation method. In addition, Mises [4] points out that, at the point $s = l/2$, the bar has the maximum deflection $y_m = c - c_1$, which, because of the smallness of the coefficient c_1 with respect to c, is approximately equal to

$$y_m = c = \frac{\sqrt{8}\,l}{\pi}\sqrt{\frac{P}{P_*} - 1}. \tag{7}$$

On the Errors in Papers on the Theory of Elastic Bars. In [6, p. 561], the coefficient c_1 is given with the wrong sign: $c_1 = c(P/P_* - 1)/8$. In [7, p. 174; 8, p. 17; 9, p. 74; 10, p. 86; 11, p. 129], this error led to the erroneous formula for the maximum deflection as a function of load:

$$f = \frac{2\sqrt{2}\,l}{\pi}\sqrt{\frac{P}{P_*} - 1}\,\Big(1 - \frac{1}{8}\Big(\frac{P}{P_*} - 1\Big)\Big); \tag{8}$$

however, the relations $y_m = c - c_1$ and $cc_1 < 0$ and the expressions for c and c_1 imply that

$$f = \frac{2\sqrt{2}\,l}{\pi}\sqrt{\frac{P}{P_*} - 1}\,\Big(1 + \frac{1}{8}\Big(\frac{P}{P_*} - 1\Big)\Big). \tag{9}$$

It should be noted that formula (9) is less accurate than formula (7), and the unjustified formula (8) is more accurate than formula (7).

In [12, p. 25], Dinnik erroneously gives the expression $c_1 = c\sqrt{P/P_* - 1}/8$ for the coefficient c_1, although in [13, p. 66; 14, p. 636], he gives the expression $c_1 = -c(P/P_* - 1)/8$ and states that formula (9) gives good results.

Nikolai [1] shows that the Mises formula (7) is already contained in the more general Euler's result if only the first two terms are kept on the right side of (1). Keeping the first three terms on the right side of (1), as a second approximation, we obtain

$$\frac{f}{l} = \frac{2\sqrt{2}}{\pi}\sqrt{\frac{P}{P_*} - 1}\,\Big(1 - \frac{19}{16}\Big(\frac{P}{P_*} - 1\Big)\Big). \tag{10}$$

Next, referring to [8] and criticizing formula (8), Dinnik [1, p. 443] arrives at the conclusion that the approximate Mises conclusion, which leads to the correct first-approximation formula (7), is insufficient for obtaining the second-approximation formula (10).

A comparative analysis of similar formulas is given in [15], where the formula given for $P/P_* \approx 1$

$$\frac{f}{l} = \frac{2\sqrt{2}}{\pi}\sqrt{\frac{P}{P_*} - 1}\Big(1 - \frac{41}{64}\Big(\frac{P}{P_*} - 1\Big)\Big)$$

is less accurate than (10) and was obtained using series in [2, p. 501] . The robustness of formula (7) also shows up in [16, p. 245], where, using catastrophe theory, a similar but incorrect relation prescribing unstable postcritical behavior of the a bar was obtained:

$$P \approx P_*(1 + (3/8)(\pi f/l_1)^2)$$

TABLE 1

Deflection formula	f/l							
	$\lambda = 1.002$	$\lambda = 1.020$	$\lambda = 1.100$	$\lambda = 1.200$	$\lambda = 1.300$	$\lambda = 1.400$	$\lambda = 1.500$	$\lambda = 2.000$
(7)	0.040 263	0.127 32	0.2847	0.403	0.493	—	—	—
(8)	0.040253	0.12701	0.2811	0.393	0.475	—	—	—
(10)	0.040168	0.12430	0.2509	0.307	0.317	0.299	0.259	—
(13)	0.040144	0.12386	0.2555	0.338	0.393	0.436	0.472	—
(16)	0.040183	0.12483	0.2588	0.336	0.379	0.407	0.424	0.450
(20)	0.040168	0.12437	0.2545	0.325	0.364	0.387	0.401	0.418
Exact solution	0.040168	0.12437	0.2543	0.324	0.361	0.382	0.394	0.398

(l_1 is the distance between the ends of the bar). In the derivation of this relation, the following errors were made: 1) the expression for the total potential energy of the bar contains l instead of l_1; 2) it is assumed that the quantity l_1 does not depend on the bar deflection; 3) some terms of the same order of smallness are unreasonably omitted.

In [17, p. 356] the following approximate solution is proposed for the slightly bended equilibrium states of a bar. In the differential equation of the bar deflection curve

$$\frac{d^2y}{ds^2} \Big/ \sqrt{1 - \left(\frac{dy}{ds}\right)^2} + \frac{P}{EI}\, y = 0 \tag{11}$$

we make the change $1/\sqrt{1-(dy/ds)^2} = 1 + (dy/ds)^2/2 + 3(dy/ds)^4/8 + \ldots \approx 1 + (dy/ds)^2/2$. As a result,

$$\frac{d^2y}{ds^2}\left(1 + \frac{1}{2}\left(\frac{dy}{ds}\right)^2\right) + \frac{P}{EI}\, y = 0.$$

Assuming that, for small deflections, the bar axis shape is described by one half-wave of the sinusoid (3), and replacing the quantity $(dy/ds)^2/2$ by its average value on the interval $[0, l]$, we have

$$\frac{d^2y}{ds^2}\left(1 + \left(\frac{c\pi}{2l}\right)^2\right) + \frac{P}{EI}\, y = 0.$$

From this, in view of equality (3), it follows that $P/P_* \approx 1 + c^2\pi^2/(4l^2)$, and, hence, $c/l \approx 2\sqrt{P/P_* - 1}/\pi$. The resulting relation between the maximum bar deflection and load is rougher than formula (7).

Formula (7) can be refined using the Mises method if, making the change of the root, we pass from Eq. (11) to the approximate equation

$$\frac{d^2y}{ds^2}\left(1 + \frac{1}{2}\left(\frac{dy}{ds}\right)^2 + \frac{3}{8}\left(\frac{dy}{ds}\right)^4\right) + \frac{P}{EI}\, y = 0. \tag{12}$$

Substituting expression (3) into (12) and performing trigonometric transformations, we obtain

$$-\left(\frac{\pi}{l}\right)^2\left[\left(1 + \frac{1}{8}\left(\frac{c\pi}{l}\right)^2 + \frac{3}{64}\left(\frac{c\pi}{l}\right)^4\right) c\sin\frac{\pi s}{l} + \left(\frac{1}{8}\left(\frac{c\pi}{l}\right)^2 + \frac{9}{128}\left(\frac{c\pi}{l}\right)^4\right) c\sin\frac{3\pi s}{l}\right.$$
$$\left. + \frac{3}{128}\left(\frac{c\pi}{l}\right)^4 c\sin\frac{5\pi s}{l}\right] + \frac{P}{EI}\, c\sin\frac{\pi s}{l} = 0.$$

Next, setting the coefficient at $\sin(\pi s/l)$ to zero, we have

$$\frac{3}{64}\left(\frac{c\pi}{l}\right)^4 + \frac{1}{8}\left(\frac{c\pi}{l}\right)^2 - (\lambda - 1) = 0,$$

where $\lambda = P/P_*$. This leads to the following formula, which is more accurate than (7) and (8):

$$\frac{f}{l} = \frac{c}{l} = \frac{2}{\pi\sqrt{3}}\sqrt{\sqrt{1 + 12(\lambda - 1)} - 1}. \tag{13}$$

As shown in Table 1, for $\lambda \geq 1.1$, the latter formula is also more accurate than formula (10).

For loads $\lambda = P/P_*$, $1.002 \leqslant \lambda \leqslant 2.000$, Table 1 gives the maximum bar deflections f normalized to the bar length l calculated by the formulas indicated in the left column. The lower row of Table 1 gives the relative bar deflection f/l calculated with the specified accuracy using the complete elliptic integral of the first kind. The blanks in the table correspond to values of f/l that have no physical meaning, for example, $f/l < 0$ or $f/l > 0.5$.

Using the expansion of the roots, from (13), one can derive a number of rougher (as shown by calculations) formulas:

$$\frac{2}{\pi\sqrt{3}}\sqrt{\sqrt{1+12(\lambda-1)}-1} \approx \frac{2\sqrt{2}}{\pi}\sqrt{\frac{6(\lambda-1)-(12(\lambda-1))^2/8}{6}}$$

$$= \frac{2\sqrt{2}}{\pi}\sqrt{\lambda-1}\,\sqrt{4-3\lambda} \approx \frac{2\sqrt{2}}{\pi}\sqrt{\lambda-1}\Big(1-\frac{24}{16}(\lambda-1)\Big).$$

Use of the Bubnov–Galerkin method. As a matter of fact, Mises derived formula (7) using the Bubnov–Galerkin method. However, in refining formula (7) by means of the expression $y_m = c - c_1$, it is necessary to calculate the coefficient c_1 and, keeping terms of the same order of smallness, to refine the coefficient c. Indeed, in the differential equation (2) of the bar deflection curve

$$\frac{d^2y}{ds^2} + \frac{P}{EI}\,y\sqrt{1-\Big(\frac{dy}{ds}\Big)^2} = 0$$

replacing the root by the series expansion

$$\sqrt{1-\Big(\frac{dy}{ds}\Big)^2} = 1 - \frac{1}{2}\Big(\frac{dy}{ds}\Big)^2 - \frac{1}{8}\Big(\frac{dy}{ds}\Big)^4 - \dots$$

and rearranging the nonlinear terms to its right side, we obtain

$$\frac{d^2y}{ds^2} + \frac{P}{EI}\,y = \frac{P}{EI}\,y\Big(\frac{1}{2}\Big(\frac{dy}{ds}\Big)^2 + \frac{1}{8}\Big(\frac{dy}{ds}\Big)^4 + \dots\Big). \tag{14}$$

Substitution of expression (3) into (14) with retention of only the first two nonlinear terms on the right side yields

$$\Big(-\Big(\frac{\pi}{l}\Big)^2 + \frac{P}{EI}\Big)c\sin\frac{\pi s}{l} = \frac{P}{EI}\,c\Big[\Big(\frac{1}{8}\Big(\frac{c\pi}{l}\Big)^2 + \frac{1}{64}\Big(\frac{c\pi}{l}\Big)^4\Big)\sin\frac{\pi s}{l}$$

$$+\Big(\frac{1}{8}\Big(\frac{c\pi}{l}\Big)^2 + \frac{3}{128}\Big(\frac{c\pi}{l}\Big)^4\Big)\sin\frac{3\pi s}{l} + \frac{1}{128}\Big(\frac{c\pi}{l}\Big)^4\sin\frac{5\pi s}{l}\Big]. \tag{15}$$

Equating the coefficients at $\sin(\pi s/l)$ in (15) leads to the load–deflection relation

$$-\Big(\frac{\pi}{l}\Big)^2 + \frac{P}{EI} = \frac{P}{EI}\Big(\frac{1}{8}\Big(\frac{c\pi}{l}\Big)^2 + \frac{1}{64}\Big(\frac{c\pi}{l}\Big)^4\Big),$$

or $z^2 + z = p = (\lambda-1)/\lambda$, where $z = (c\pi/l)^2/8$ and $\lambda = P/P_* \geqslant 1$. From this, $z = (-1+\sqrt{1+4p})/2 \approx (-1+1+4p/2-(4p)^2/8)/2 = p(1-p)$. Hence,

$$\frac{f}{l} = \frac{c}{l} = \frac{\sqrt{8p(1-p)}}{\pi} = \frac{2\sqrt{2}}{\pi\lambda}\sqrt{\lambda-1}. \tag{16}$$

We note that in [18] formula (16) was obtained using a different method (quadratic approximation of the exact solution) and differs from (7) only by the factor λ in the denominator.

Thus, assuming that the bar takes the shape of the half-wave of the sinusoid (3), and ignoring the coefficient c_1, we obtain formula (16), which is more accurate than (7) and (8) and which, for $\lambda \geqslant 1.2$ (see Table 1), is also more accurate than formulas (10) and (13). Furthermore, formula (16) proves more accurate than formula (13) although the representation of the root in the original differential equation and the representation of the inverse root in Eq. (11) contain the same number of series terms. The main reason for this difference is as follows. Because, for $z \neq 0$,

$$(1+z^2/2+3z^4/8)(1-z^2/2-z^4/8) = 1 - z^6/4 - 3z^8/64 < 1,$$

it follows that for $0 < |z| < 1$, $(1+z^2/2+3z^4/8) < 1/(1-z^2/2-z^4/8)$. Hence,

$$0 < \frac{1}{\sqrt{1-z^2}} - \frac{1}{1-z^2/2-z^4/8} < \frac{1}{\sqrt{1-z^2}} - (1+z^2/2+3z^4/8),$$

$$\sqrt{1-z^2} - \frac{1}{1+z^2/2+3z^4/8} < \sqrt{1-z^2} - (1-z^2/2-z^4/8) < 0.$$

Therefore, in both the numerator and denominator, it is reasonable to replace the root $\sqrt{1-z^2}$ by the expression $1-z^2/2-z^4/8$ and not by the expression $1/(1+z^2/2+3z^4/8)$. In the case considered, it can be assumed that the relation

$$\frac{d^2y}{ds^2}+\frac{P}{EI}\,y=\frac{P}{EI}\,y\Big(\frac{1}{2}\Big(\frac{dy}{ds}\Big)^2+\frac{1}{8}\Big(\frac{dy}{ds}\Big)^4\Big) \tag{17}$$

is a more accurate approximation of the original equation (11) than Eq. (12).

To refine formula (16), we will assume that the bar takes the shape described by a linear combination of two sinusoids with successive odd numbers of half-waves:

$$y=c\sin(\pi s/l)+d\sin(3\pi s/l). \tag{18}$$

The differential operator $(\cdot)((\cdot)'^2/2+(\cdot)'^4/8)$ transforms (18) to the expression

$$y\Big(\frac{y'^2}{2}+\frac{y'^4}{8}\Big)=\Big(\frac{\pi^2}{8l^2}\,(c^2-5cd+18d^2)+\frac{\pi^4}{128l^4}\,(2c^4-9c^3d+108c^2d^2-270cd^3+486d^4)\Big)c\sin\frac{\pi s}{l}$$
$$+\Big(\frac{\pi^2}{8l^2}\,(c^3+2c^2d+9d^3)+\frac{3\pi^4}{128l^4}\,(2c^5+2c^4d+22c^3d^2+36c^2d^3+54d^5)\Big)\sin\frac{3\pi s}{l},$$

in which the linear combination of sinusoids with numbers of half-waves larger than three is omitted. Thus, substituting expression (18) into (17) and equating the coefficients at $\sin(\pi s/l)$ and $\sin(3\pi s/l)$, we obtain a system of nonlinear algebraic equations for the coefficients c and d for a given load P. Denoting $a=\pi c/l$, $b=\pi d/l$, and $\lambda=P/P_*$, we have the following system of equations to determine a and b in terms of λ:

$$\frac{\lambda-1}{\lambda}=\frac{a^2-5ab+18b^2}{8}+\frac{2a^4-9a^3b+108a^2b^2-270ab^3+486b^4}{128},$$
$$\frac{\lambda-9}{\lambda}\,b=\frac{a^3+2a^2b+9b^3}{8}+3\frac{a^5+2a^4b+22a^3b^2+36a^2b^3+54b^5}{128}.$$

Eliminating λ from the second equation and omitting terms of higher order of smallness, we obtain

$$\begin{gathered}a^4/8+18b^2-5ab+a^2-8p=0,\\ a^3-7a^2b+45ab^2-153b^3+64b=0,\end{gathered} \tag{19}$$

where $p=(\lambda-1)/\lambda$. Substituting the approximate expression $b\approx -a^3/64$, which follows from the second equation, into the first equation of system (19), we obtain the relation $a^4/8+18(a^3/64)^2+5a^4/64+a^2-8p=0$, which, in turn, leads to the approximate equation $13a^4+64(a^2-8p)=0$. From this, we have $a^2=(-32+32\sqrt{1+13p/2})/13\approx 8p-13p^2$, and $a\approx 2\sqrt{2p}\,(1-13p/16)$ and, hence,

$$a-b\approx a(1+a^2/64)\approx 2\sqrt{2p}\,(1-13p/16)(1+p/8)\approx 2\sqrt{2p}\,(1-11p/16),$$

i.e., the maximum relative deflection of the bar is approximately equal to

$$\frac{f}{l}=\frac{c-d}{l}=\frac{2\sqrt{2}}{\pi}\sqrt{\frac{\lambda-1}{\lambda}}\Big(1-\frac{11}{16}\,\frac{\lambda-1}{\lambda}\Big). \tag{20}$$

Calculations have shown (see Table 1) that, in the range $1\leqslant\lambda\leqslant 2$, formula (20) proves to be the most accurate among the approximate formulas given here. The maximum relative error of formula (20) takes place for $\lambda=2$ and does not exceed 5%, which is quite admissible from the engineering point of view. For a load $\lambda\approx 1.75$, the bar deflection reaches the maximum $f_{\max}\approx 0.403$, and, for $\lambda\approx 2.18$, the bar forms a loop. It should be noted, however, that formula (16) provides a better qualitative description of the load–deflection relation than formula (20). Indeed, calculating the deflection by formula (16), we obtain $f/l\to 0$ as $\lambda\to\infty$, which, in contrast to the relation $f/l\to 5\sqrt{2}/(8\pi)\neq 0$ as $\lambda\to\infty$ for formula (20), is in line with the mechanical content of the problem.

Figure 1 shows the exact curve of the relative deflection of the middle of the bar f/l versus load λ (curve 1) and the curves constructed using approximate formulas.

Buckling a Three-Layer Bar. The nonlinear differential equilibrium equation for the deflection curve of a longitudinally compressed hinged three-layer bar is written as [19, p. 47]

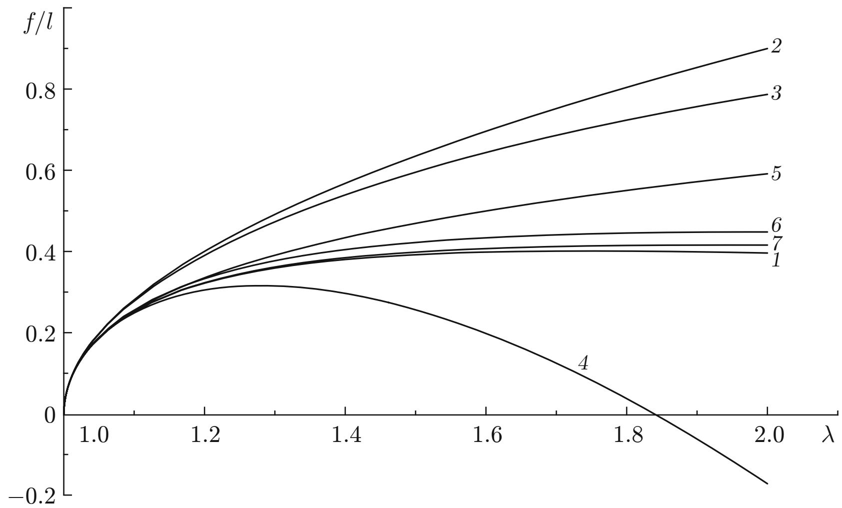

Fig. 1. Relative bar deflection f/l versus load λ: curve 1 refers to the exact solution; curves 2–7 refer to the solutions obtained by approximate formulas (7) (curve 2), (8) (curve 3), (10) (curve 4), (13) (curve 5), (16) (curve 6), and (20) (curve 7).

$$\Big(1 - ht\,\frac{d^2}{ds^2}\Big)\frac{d^2}{ds^2}\Big(\frac{d^2y}{ds^2}\Big/\sqrt{1-\Big(\frac{dy}{ds}\Big)^2}\Big) + \frac{N}{D}\Big(1 - h\,\frac{d^2}{ds^2}\Big)\frac{d^2y}{ds^2} = 0, \qquad (21)$$

where N is the load parameter and $y(s)$ is the bar deflection; the parameters D (minimum flexural rigidity of the bar), h, and t completely characterize the structure of the three-layer bar in bending [19, p. 21]. We note that, if the thickness of the filler layer is equal to zero (for a single-layer bar), $h = 0$ and Eq. (21) coincides with the twice-differentiated equation (11). Expanding the inverse root in (21) in a series and retaining first three terms of this series, we obtain

$$\Big(1 - ht\,\frac{d^2}{ds^2}\Big)\frac{d^2}{ds^2}\Big\{\frac{d^2y}{ds^2}\Big[1+\frac{1}{2}\Big(\frac{dy}{ds}\Big)^2+\frac{3}{8}\Big(\frac{dy}{ds}\Big)^4\Big]\Big\} + \frac{N}{D}\Big(1 - h\,\frac{d^2}{ds^2}\Big)\frac{d^2y}{ds^2} = 0. \qquad (22)$$

Substituting expression (3) into (22), performing trigonometric transformations, and setting the coefficient at $\sin(\pi s/l)$ to zero, we have

$$3k^3(1+kht)c^4 + 8k^2(1+kht)c^2 + 64k(1+kht) - 64(1+kh)N/D = 0$$

or

$$3k^2c^4 + 8kc^2 - 64(\lambda - 1) = 0.$$

Here $k = (\pi/l)^2$ and $\lambda = N/N_*$, where $N_* = Dk(1+kht)/(1+kh)$ is the Eulerian critical force [19]. Finally, for the maximum relative deflection f/l, we obtain, in essence, the formula for homogeneous bars (13)

$$\frac{f}{l} = \frac{c}{l} = \frac{2}{\pi\sqrt{3}}\sqrt{\sqrt{1+12(\lambda-1)}-1},$$

which refines formula (1.219) from [19, p. 48]. In this formula, however, the Eulerian load N_* depends on the parameters characterizing the shear of the filler and the structure of the bar.

Assuming that the shape of the bar axis is described by a linear combination of two sinusoids (18) with successive odd numbers of half-waves, we have the following system of two nonlinear algebraic equations to determine the coefficients c and d from the given load N:

$$\begin{gathered} 3a^4 + 8a^2 + 24ab - 64(\lambda - 1) = 0, \\ 9a^5 + 16a^3 + 288a^2b + 1152b - 128b\lambda\,\frac{(1+kht)(1+9kh)}{(1+kh)(1+9kht)} = 0. \end{gathered} \qquad (23)$$

Here $a = \pi c/l$, $b = \pi d/l$, $\lambda = N/N_*$, $N_* = Dk(1+kht)/(1+kh)$, and $k = (\pi/l)^2$. Assuming that, in the second equation of system (23), the fraction is approximately equal to 1, $\lambda \approx 1$, and omitting terms of higher order of smallness than a^3, we have the approximate equation $16a^3 + 1152b - 128b = 0$. This leads to the expression $b \approx -a^3/64$, whose substitution into the first equation of system (23) yields $21a^4 + 64a^2 - 512(\lambda - 1) = 0$. Thus,

$$a^2 = \frac{16\sqrt{2}}{21}\,(\sqrt{21\lambda - 19} - \sqrt{2}) \approx \frac{16\sqrt{2}}{21}\,\sqrt{2}\Big(\frac{21}{4}\,(\lambda - 1) - \frac{21^2}{32}\,(\lambda - 1)^2\Big) = 8(\lambda - 1) - 21(\lambda - 1)^2.$$

In this case, for the maximum relative deflection of the bar,

$$\frac{f}{l} = \frac{c - d}{l} = \frac{a - b}{\pi} \approx \frac{a(1 + a^2/64)}{\pi}$$

$$\approx \frac{2\sqrt{2}\,\sqrt{\lambda - 1}}{\pi}\sqrt{1 - \frac{21}{8}\,(\lambda - 1)}\,\Big(1 + \frac{1}{8}\,(\lambda - 1)\Big) \approx \frac{2\sqrt{2}\,\sqrt{\lambda - 1}}{\pi}\Big(1 - \frac{19}{16}\,(\lambda - 1)\Big).$$

Thus, if all terms of the same order of smallness are taken into account, the maximum deflection of a three-layer bar is described by the formula for homogeneous bars (10). We note that for a single-layer bar ($h = 0$), the previous calculations are significantly simplified, in particular, in the second equation of system (23), the fraction is exactly equal to unity.

Conclusions. Using the Bubnov–Galerkin method to solve nonlinear differential equations leads to significant difficulties related to the necessity of solving systems of nonlinear algebraic equations. However, the description of the subcritical behavior of a compressed elastic bar shows that, with an appropriate choice of the coordinate functions for the representation of the solution and with a correct simplification of the nonlinear algebraic equations with all quantities of the same order of smallness taken into account, the Bubnov–Galerkin method yields more accurate formulas than those obtained using different methods. Thus refined expressions for the maximum deflection of a three-layer bar were obtained.

REFERENCES

1. E. L. Nikolai, "On Euler's papers on buckling theory," in: *Papers on Mechanics* [in Russian], Gostekhteoretizadt, Moscow (1955).
2. A. N. Krylov, "Equilibrium shapes of compressed slabs under buckling," in: *A. N. Krylov. Selected Paper* [in Russian], Izd. Akad. Nauk SSSR, Moscow (1958).
3. E. P. Popov, *Theory and Calculation of Elastic Bars* [in Russian], Nauka, Moscow (1986).
4. R. Mises, "Ausbiegung eines auf Knicken beanspruchten Stabes", *Z. Angew Math. Mech.*, **4**, 435–436 (1924).
5. N. S. Astapov, "Postcritical behavior of a bar," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 92, Inst. of Hydrodynamics, Sib. Div., Acad. of Sci. of the USSR, Novosibirsk (1989), pp. 14–21.
6. S. P. Timoshenko, "Problems of elastic stability," in: E. I. Grigolyuk (ed.), *Stability of Bars, Plates, and Shells* [in Russian], Nauka, Moscow (1971).
7. S. P. Timoshenko, *Resistance of Materials* [in Russian], Part 2, Gostekhteoretizdat, Leningrad–Moscow (1932).
8. S. P. Timoshenko, *Stability Problems of Elastic Systems* [in Russian], Kubuch, Leningrad (1935).
9. S. P. Timoshenko, *Stability of Elastic Systems* [in Russian], OGIZ, Moscow–Leningrad (1946).
10. S. P. Timoshenko, *Stability of Elastic Systems* [in Russian], 2nd ed., Gostekhteoretizdat, Moscow (1955).
11. S. P. Timoshenko, *Resistance of Materials* [in Russian], Vol. 2, Nauka, Moscow (1965).
12. *A. N. Dinnik. Selected Papers* [in Russian], Vol. 3, Izd. Akad. Nauk USSR (1956).
13. A. N. Dinnik, *Buckling* [in Russian], GONTI, Moscow–Leningrad (1939).
14. A. N. Dinnik (ed.), *Handbook on Technical Mechanics* [in Russian], OGIZ–Gostekhteoretizdat, Moscow (1949).
15. N. S. Astapov, "Approximate formulas for deflection of compressed elastic bars," *J. Appl. Mekh. Tech. Fiz.*, **37**, No. 4, 573–576 (1996).
16. R. Gilmore, *Catastrophe Theory for Scientists and Engineers*, Wiley, New York (1981).
17. I. G. Teregulov, *Resistance of Materials and Fundamentals of the Theory of Elasticity and Plasticity* [in Russian], Vysshaya Shkola, Moscow (1984).
18. I. S. Astapov, N. S. Astapov, and E. L. Vasil'eva, "Quadratic approximation of large displacements of an elastic compressed bar," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 1, 164–171 (2003).
19. É. I. Grigolyuk and P. P. Chulkov, *Stability and Vibrations of Three-Layer Shells* [in Russian], Mashinostroenie, Moscow (1973).